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Rings of Operators on Modules over Commutative Rings and Their Right Ideals

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Suppose that there is an inclusion of k-algebras $R \subseteq E \subseteq \operatorname{End}_k M$ with R commutative and E non-commutative. We introduce and impose conditions under which the finitely generated essential right ideals of E may be classified in terms of k-submodules of M. This yields a classification of the domains Morita equivalent to E when E is a Noetherian domain. For example, a special case of our results is:

THEOREM. *Let R be a commutative Noetherian k-algebra which is domain. Let E be a simple Ore extension of R of the form $R[x, x^{-1}; \sigma]$ or $R[x; \delta]$ (in the latter case we must also assume $R \supset \mathbb{Q}$). Then, for a certain sublattice $\mathscr{L}$ of the lattice of k-submodules of R:*

(a) *Every non-zero right ideal of E is isomorphic to one of the form*

$$E(R, V) := \{\theta \in E : \theta(R) \subseteq V\},$$

for some $V \in \mathscr{L}$.

(b) *Every domain Morita equivalent to E is isomorphic to*

$$E(V) := \{\theta \in E \otimes \operatorname{Frac} R : \theta(V) \subseteq V\},$$

for some $V \in \mathscr{L}$. Conversely, if R is Dedekind, then $E(V)$ is Morita equivalent to E, for $V \in \mathscr{L}$.

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INTRODUCTION

In [SS] Smith and Stafford found some remarkable examples of domains Morita equivalent to $\mathscr{D}(R)$, the ring of differential operators on the coordinate ring R of a smooth affine curve. They proved that $\mathscr{D}(R) \overset{\mathrm{M}}{\sim} \mathscr{D}(S)$, for S a subalgebra of R with integral closure R and such that $\operatorname{Spec} R \to \operatorname{Spec} S$ is injective. One of their main techniques was to use the right ideal $\mathscr{D}(R, S) = \{\theta \in \mathscr{D}(R) : \theta(R) \subseteq S\}$ of $\mathscr{D}(R)$. In [CH1] Cannings and Holland showed that by widening the possibilities in the second argument of $\mathscr{D}(R, \_)$ it was possible to classify the right ideals of $\mathscr{D}(R)$ up to isomorphism and, further, to classify the domains Morita equivalent to $\mathscr{D}(R)$. In this paper we extend the methods of [CH1] (and [CH2]) to more general rings of operators. For simplicity, in the Introduction we will just consider two main examples. Let R be a commutative Noetherian k-algebra which is a domain. We are interested in the rings E which are simple Ore extensions of R of the form $R[x, x^{-1}; \sigma]$ or $R[x; \delta]$ (in the latter case assume $R \supset \mathbb{Q}$).

Write $\mathscr{I}(E_E)$ for the lattice of right ideals of E which contain a non-zero ideal of R and $\mathscr{I}(R_k)$ for the lattice of k-submodules of R which contain some non-zero ideal of R. We may pass back and forth between $\mathscr{I}(E_E)$ and $\mathscr{I}(R_k)$. Indeed, given $V \in \mathscr{I}(R_k)$ and $D \in \mathscr{I}(E_E)$, define

$$E(R, V) := \{\theta \in E : \theta * R \subseteq V\} \qquad \text{and}$$

$$D * R := \left\{ \sum_i d_i * r_i : d_i \in D, r_i \in R \right\},$$

where $*$ denotes evaluation. The non-zero elements of R are an Ore set in E and so one can form the localization $E \otimes \operatorname{Frac} R$. This acts naturally on $\operatorname{Frac} R$ and so, if $V \in \mathscr{I}(R_k)$, one can consider the subring:

$$E(V) := \{\theta \in E \otimes \operatorname{Frac} R : \theta * V \subseteq V\}.$$

The first main step is applying the general results of this paper to the special case of E is to show that $\mathscr{I}(E_E)$ is faithfully represented in $\mathscr{I}(R_k)$ (see 5.4, 5.5).

CORRESPONDENCE THEOREM. *Let R be a commutative Noetherian k-algebra which is a domain. Let E be a simple Ore extension of R of the form $R[x, x^{-1}; \sigma]$ or $R[x; \delta]$* (*in the latter case assume $R \supset \mathbb{Q}$*). *Then the map $\_ * R : \mathscr{I}(E_E) \to \mathscr{I}(R_k)$ is injective and $D = E(R, D * R)$, whenever $D \in \mathscr{I}(E_E)$.*

We can now state our main theorem (see Theorems 2.10, 2.11, 2.13) which classifies the right ideals of E and the domains Morita equivalent to E.

MAIN THEOREM. *Let R be a commutative Noetherian k-algebra which is a domain. Let E be a simple Ore extension of R of the form $R[x, x^{-1}; \sigma]$ or $R[x; \delta]$ (in the latter case assume $R \supset \mathbb{Q}$).*

(a) *If D is a non-zero right ideal of E then $D \cong E(R, V)$, as right E-modules, for some $V \in \mathscr{I}(E_E) * R$.*

(b) *If S is a domain Morita equivalent to E then S is isomorphic to $E(V)$, for some $V \in \mathscr{I}(E_E) * R$. Conversely, if R is Dedekind, then $E(V)$ is Morita equivalent to E whenever $V \in \mathscr{I}(E_E) * R$.*

Having discussed our two main examples we complete the Introduction by explaining, briefly, the more general point of view which we take in the remainder of the paper. Let E be a non-commutative k-algebra, which (to simplify things here) is a domain. Suppose that E is an algebra of operators on a faithfully flat module M over a commutative Noetherian domain R. Thus, $k \subseteq R \subseteq E \subseteq \operatorname{End}_k M$. Again, we aim to classify the right ideals of E and the domains Morita equivalent to E, but, now, in terms of k-submodules of M. In order to do this, one has to impose hypotheses. Specifically, one assumes that:

(1) (similarly to the Correspondence Theorem, above) $\mathscr{I}(E_E)$ is faithfully represented in $\mathscr{I}(E) * M$;

(2) the non-zero elements of R are an Ore set of E and the localization is a PID.

With these two hypotheses one can obtain a similar result to the Main Theorem. The more general setting allows one to obtain additional examples. For instance, our results apply to the "quantum" Weyl algebra and idealizers like $k + xA_1(k)$.

We briefly review the contents of this paper. Section 1 develops criteria for the correspondence. Section 2 discusses localization of E at the set of S of regular elements of R, giving a condition for localization to be possible, proving that $S^{-1}E$ acts on $S^{-1}R$ and proving the Main Theorem. Section 3 discusses a left module version of the correspondence. Section 4 discusses further the bimodules $E(V, W)$, the tensor products of such bimodules, the homomorphisms between them, and the minimal subbimodules. It considers the structure of $E(V)$, for more general V. Section 5 gives examples that satisfy (1) and (2) and discusses the shape of $\mathscr{I}(E_E) * M$.

For the convenience of the reader we list all the main notations that are used in the paper and where they first occur:

(1.1) $D * M$, $E(M, V)$, $\mathscr{L}(N_A)$;

(1.2) Φ_D;

(1.4) $k, R, M,$ (B);

(1.5) $\mathscr{I}(M_k), \mathscr{R}(M_k), \mathscr{I}(E_E), \mathscr{R}(E_E)$, (C), (C$'$);

(2.1) S;

(2.5) $E(V,W), E(V)$;

(2.6) $\hat{\mathscr{I}}(M_k), \hat{\mathscr{R}}(M_k), \hat{\mathscr{I}}(E_E), \hat{\mathscr{R}}(E_E)$;

(2.8) (L);

(2.14) (G);

(3.1) $\hat{\mathscr{P}}(E_E), \hat{\mathscr{I}}({}_E E), \hat{\mathscr{P}}({}_E E)$;

(3.2) $D *^{-1} M$;

(3.3) (FL);

(4.1) V^+, V^-;

(4.8) $J(V,W)$.

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1. THE CORRESPONDENCE FOR RIGHT IDEALS

1.1. We begin this section with some rather general observations. Suppose that E is a ring and M is a left E-module. If D is a right ideal of E write $D * M := DM$; it is a right $\operatorname{End}_E M$-submodule of M. On the other hand, if V is any additive subgroup of M denote by $E(M,V)$ the corresponding right ideal of E. By definition, $E(M,V) = \{\theta \in E : \theta M \subseteq V\}$.

If ${}_E M$ is a progenerator then $\_ * M : \mathscr{L}(E_E) \to \mathscr{L}(M_{\operatorname{End}_E M})$ gives a lattice isomorphism between the lattice of right ideals of E and the lattice of right $\operatorname{End}_E M$-submodules of M with the inverse given by $E(M, \_)$. (For any module N over a ring A we write $\mathscr{L}(N_A)$ for its lattice of A-submodules.) When ${}_E M$ is no longer a progenerator one cannot expect such a result. However, for many interesting examples there exists a sublattice $\mathscr{L}$ of the lattice of right ideals of E such that $\_ * M$: $\mathscr{L} \to \mathscr{L}(M_{\operatorname{End}_E M})$ is an injective lattice map. We shall shortly investigate conditions under which this occurs. But first let us point out the usefulness of such a result with an example (taken from [CH1]). In the example ${}_E M$ is far from being a progenerator; it is a simple module, and $\operatorname{End}_E M$ is very small but $\mathscr{L}$ is big enough to yield useful information about E.

EXAMPLE. Let k be an algebraically closed field of characteristic zero; R a Dedekind domain, finitely generated as a k-algebra; and $\mathscr{D}(R)$ the ring of k-linear differential operators on R.

Then, if one takes $E = \mathscr{D}(R)$, $M = R$, and $\mathscr{L}$ to be the set of right ideals of $\mathscr{D}(R)$ that contain a non-zero ideal of R one has that $\_ * M$: $\mathscr{L} \to \mathscr{L}(M_k)$ is an injective lattice map. In particular, if $D \in \mathscr{L}$ then $D = E(M, D * M)$. Furthermore, every non-zero right ideal of $\mathscr{D}(R)$ is isomorphic to one in $\mathscr{L}$. There is also a very succint description of the lattice $\mathscr{L} * M$, see Section 5.

1.2. Let us introduce general conditions under which one can prove such a result. So suppose that E is a ring, M is a left E-module, and $\mathscr{L}$ is a subset of the lattice of right ideals of E that is closed under $+$ and contains E. We consider the conditions:

(1.2.1) If $D' \supsetneqq D$ with $D, D' \in \mathscr{L}$ then $(D'/D) \otimes_E M \neq 0$.

(1.2.2) If $E \supsetneqq D$ with $D \in \mathscr{L}$ then $\mathrm{Tor}_1^E(E/D, M) = 0$.

(1.2.2′) If $D \in \mathscr{L}$ then the natural map Φ_D: $D \otimes_E M \to D * M$ is an isomorphism.

(1.2.3) If $D \in \mathscr{L}$ and $D' \supseteq D$, for a right ideal D' of E, then $D' \in \mathscr{L}$.

1.3. THEOREM. (a) *Conditions* (1.2.2) *and* (1.2.2′) *are equivalent.*

(b) *Suppose that* (1.2.2) *holds. Then* (1.2.1) *holds if and only if* $\_ * M$: $\mathscr{L} \to \mathscr{L}(M_{\mathrm{End}_M})$ *is an injective map.*

(c) *If* (1.2.1), (1.2.2), *and* (1.2.3) *hold then* $D = E(M, D * M)$, *for any* $D \in \mathscr{L}$.

(d) *If* (1.2.1) *and* (1.2.2) *hold and* $\mathscr{L}$ *is a lattice then* $\_ * M$ *is a lattice map.*

Proof. (a) Consider $D \subsetneqq E$ with $D \in \mathscr{L}$. Applying $\_ \otimes_E M$ to the exact sequence $0 \to D \to E \to E/D \to 0$ we obtain the following commuting diagram with surjective vertical maps:

$$\begin{array}{ccccccc} 0 = \mathrm{Tor}_1^E(E, M) & \longrightarrow & \mathrm{Tor}_1^E(E/D, M) & \longrightarrow & D \otimes_E M & \xrightarrow{i} & E \otimes_E M \\ & & & & {\scriptstyle \Phi_D}\big\downarrow & & {\scriptstyle \Phi_E}\big\downarrow \\ & & & & D * M & \xrightarrow{\subseteq} & M \end{array}$$

As Φ_E is an isomorphism, we see that Φ_D is an isomorphism if and only if i is injective. But this injectivity is certainly equivalent to $\mathrm{Tor}_1^E(E/D, M) = 0$. Thus, we have the equivalence of (1.2.2) and (1.2.2′).

(b) Suppose that we have $D \subsetneqq D'$ for right ideals $D, D' \in \mathscr{L}$. Then, by (1.2.2′), the maps Φ_D and $\Phi_{D'}$ are isomorphisms. Thus, we obtain an exact sequence:

$$0 \to D * M \overset{\subseteq}{\to} D' * M \to (D'/D) \otimes_E M \to 0.$$

The right-hand factor is non-zero if and only if $D * M \neq D' * M$. In particular, if $\_ * M$ is injective then (1.2.1) holds. On the other hand, suppose that $D, D' \in \mathscr{L}$ and that $D * M = D' * M$. Clearly $D * M = (D + D') * M = D' * M$. Thus, applying the above argument to the containments $D \subseteq D + D' \supseteq D'$ we obtain that $D = D'$.

(c) Since $D * M = E(M, D * M) * M$, we cannot possibly have $D \subsetneqq E(M, D * M)$.

(d) Next we show that $\_ * M$ is a lattice map. It clearly preserves addition of modules. It remains to show that it preserves intersection. Suppose then that we are given $D, D' \in \mathscr{L}$ and apply $\_ \otimes_E M$ to the exact sequence

$$0 \to D \cap D' \to D \oplus D' \to D + D' \to 0.$$

Recalling that $\Phi_{D \cap D'}$, Φ_D, $\Phi_{D'}$, and $\Phi_{D+D'}$ are all isomorphisms we obtain the exactness of

$$(D \cap D') * M \to D * M \oplus D' * M \to (D' + D) * M = D' * M + D^* M \to 0.$$

That is, $D * M \cap D' * M = (D \cap D') * M$.

1.4. Our problem then is to find workable criteria under which the conditions of 1.2 hold true. Of course, they are true in the "trivial case" when M is a progenerator, with $\mathscr{L} = \mathscr{L}(E_E)$. More interestingly they are true with E, M, and $\mathscr{L}$ as in Example 1.1. We will review this particular example in more detail in Section 5. In fact, this example is typical of the sort we have in mind: a ring of operators on a module over a commutative algebra.

We now formulate our basic assumptions and notation that will be *in force throughout the paper*. Any other hypothesis will always be additional to these.

BASIC HYPOTHESIS (B). *We suppose that we have the following inclusions of k-algebras*: $k \subseteq R \subseteq E \subseteq \operatorname{End}_k M$, *where k is a commutative ring*, R *is a commutative ring, and M is faithfully flat as an R-module*.

Remark. (a) E is a torsionfree left R-module.

(b) If R is a Dedekind domain then E is a flat left R-module.

Proof. Recall that for any ring A, an A-module N is said to be torsionfree if $an = 0$, with $a \in A$ regular and $n \in N$, implies that $n = 0$. Note that (b) is a consequence of (a) and so we just prove (a). If $\phi \in E$ and $s \in R$ is regular then $s\phi = 0$ implies that $s(\phi * M) = (s\phi) * M = 0$. Since M is flat as an R-module, multiplication by s is injective and this means $\phi * M = 0$; that is, $\phi = 0$.

1.5. We need to introduce some notation. Let $\mathscr{I}(M_k) := \{V$ a k-submodule of M such that $V \supseteq IM$, for a non-zero ideal I of $R\}$ and let $\mathscr{R}(M_k)$ denote the sublattice of $\mathscr{I}(M_k)$ consisting of those V such that $V \supseteq sM$, for some regular element of $s \in R$. Further, $\mathscr{I}(E_E)$ denotes the set of right ideals of E that contain a non-zero ideal of R and $\mathscr{R}(E_E)$ denotes the sublattice of right ideals that contain a regular element of R.

In this section of the paper we are concerned criteria under which the conclusion of 1.3(b) holds for $\mathscr{L} = \mathscr{I}(E_E)$. In subsequent sections we will exploit the consequences of this. For clarity let us restate 1.3(b) in this special case.

The Correspondence for Right Ideals (C). The evaluation map $\_ * M$: $\mathscr{I}(E_E) \to \mathscr{I}(M_k)$ is injective.

Remark. Note that if (C) holds then

(a) $D = E(M, D * M)$, for $D \in \mathscr{I}(E_E)$;

(b) $E(M, IM) = IE$, for a non-zero ideal of E.

It will sometimes be convenient to work with the lattice $\mathscr{R}(E_E)$. For this reason we shall also consider a stronger version of the correspondence (C′) for $\mathscr{R}(E_E)$. Note that (C′) ⇒ (C) when R is a domain. In certain restricted circumstances (C) ⇒ (C)′ (see Remark 3.1, below).

The Correspondence for Right Ideals′ (C′). The evaluation map $\_ * M$: $\mathscr{R}(E_E) \to \mathscr{R}(M_k)$ is an injective lattice map.

1.6. Let us observe another consequence of the correspondence.

LEMMA. *Suppose that* (C) *holds and let I be be a non-zero ideal of R. Then there is an injection of k-algebras*, $\operatorname{End}(E/IE)_E \hookrightarrow \operatorname{End}_k(M/IM)$.

Proof. Denote the idealizer of IE_E by $\mathbb{I}(IE)$. Observe that if $\theta \in \mathbb{I}(IE)$ then $\theta * IM = (\theta IE) * M \subseteq IE * M = IM$. We have

$$\operatorname{End}(E/IE)_E \cong \mathbb{I}(IE)/IE = \mathbb{I}(IE)/E(M, IM) \hookrightarrow \operatorname{End}_k(M/IM).$$

1.7. We now introduce a condition under which the correspondence holds.

Filtration Condition (F). (a) Whenever I is a non-zero ideal of R and N is a non-zero subfactor of E/IE then N has a chain of submodules:

$$N = N_0 \supset N_1 \supset \cdots \supset N_t = 0$$

such that each factor $N_i/N_{i+1} \cong E/P_iE$, for some $P_i \in \operatorname{Spec} R$ containing I.

(b) E is flat as a left R-module.

Remark. (a) If R is a Dedekind domain and E/mE is simple for all maximal ideals m of R then (F) holds.

(b) If (F) holds and E is right Noetherian then E/mE is simple for all maximal ideals m of R.

Proof. (a) Note that $(R/m) \otimes_R E \cong E/mE$. Now R/I has a chain in which the factors have the form R/m, so E/IE has finite length and has composition factors of the form E/mE, using the flatness of ${}_RE$ (1.4).

(b) Suppose that there is a right ideal D with $E \supsetneqq D \supsetneqq mE$. By hypothesis, E/D has a chain with factors of the form E/mE. It follows that E/mE is a proper factor of itself.

1.8. Before we show that (F) $\Rightarrow$ (C) let us briefly note another simple consequence of (F). The proof is left to the reader.

LEMMA. *Suppose that* (F) *holds, and that R is Noetherian. Then every right ideal of E containing a non-zero ideal of R is finitely generated. If, in addition, R is a Dedekind domain then every such right ideal is projective.*

1.9. We now come to the main result of this section.

THEOREM. (F) $\Rightarrow$ (C) *and* (C′).

Proof. We must establish the conditions of 1.2. For then we may apply Theorem 1.3. Note that (1.2.3) is automatic for $\mathscr{I}(E_E)$. Just assuming (B) and that ${}_RE$ is flat one has that then Φ_{IE} is an isomorphism for any ideal I of R. For,

$$\begin{aligned} IE \otimes_E M &\cong (I \otimes_R E) \otimes_E M \\ &\cong I \otimes_R (E \otimes_E M) \cong I \otimes_R M \cong IM. \end{aligned}$$

If $D' \supsetneqq D$ are right ideals in $\mathscr{I}(E_E)$ then, by (F), D'/D has E/PE as a factor, for some prime ideal of R. Now $(E/PE) \otimes_E M \cong M/PM$. This latter k-module is non-zero since ${}_RM$ is faithfully flat. Now tensor product is right exact and so it follows that $(D'/D) \otimes_E M$ has the non-zero quotient $(E/PE) \otimes_E M$. Thus, (1.2.1) is verified.

Finally, (F) together with the long exact sequence for Tor makes it clear that to verify that $\operatorname{Tor}_1^E(D'/D, M) = 0$ it is enough to check that $\operatorname{Tor}_1^E(E/PE, M) = 0$. But Φ_{PE} is an isomorphism and so this follows from the Proof of Theorem 1.3(a).

2. LOCALIZATION

2.1. Recall from 1.4 that we make the basic hypothesis (B). Denote by S the set of regular elements of R. We begin this section with a discussion of the conditions required to allow localization at S and of the nature of the localizing process. It transpires that the left Ore condition and right Ore condition for S both have their own particular rôle to play. We begin with the right Ore condition. The next result shows that the condition that S be a right Ore set amounts to each of the operators in E satisfying a "continuity" condition. Since the collection of all endomorphisms of M that satisfy this condition form a ring, checking that S is a right Ore set can be reduced to checking that a generating set for E satisfies this "continuity" condition. It is perhaps worth observing that multiplication by elements of R and (in the case $M = R$) ring automorphisms of R, differential operators, and σ-derivations all satisfy this condition.

LEMMA. *Suppose that* (C) *holds. Then the following two conditions on E are equivalent.*

(1) *The subset S is a right Ore set of E.*

(2) *For every $\theta \in E$ and every ideal I of R containing a regular element of R there exists an ideal J of R containing a regular element with $\theta * JM \subseteq IM$.*

Remark. If the equivalent conditions of the proposition hold then S will be a right denominator set, since it consists of right regular elements of E by 1.4.

Proof. Suppose S is a right Ore set. Let $\theta \in E$ and I be a non-zero ideal of R containing a regular element $s \in I$. By the right Ore condition there exists $t \in S$ and $\phi \in E$ such that $\theta t = s\phi$. Therefore $\theta * tM = s\phi * M \subseteq sM \subseteq IM$. Thus (1) implies (2).

Suppose $\theta \in E$ and $s \in S$. If condition (2) holds then $\theta * tM \subseteq sM$, for some regular $t \in R$. Therefore, by (C), $\theta t \in sE$ and the right Ore condition is satisfied.

2.2. The main rôle for the right Ore condition for S is in connection with the next result.

LEMMA. *Suppose S is a right Ore set for E. Then there is an isomorphism*

$$\operatorname{Hom}(D_E, D'_E) \cong \{\theta \in ES^{-1} : \theta D \subseteq D'\},$$

whenever $D, D' \in \mathscr{R}(E_E)$.

Proof. This is similar to [MR, 3.1.15].

2.3. Now we concentrate on the left Ore condition. This is usually invoked together with the hypothesis that E is a torsionfree right R-module. So we begin with a remark about that.

LEMMA. *The following conditions are equivalent*:

(1) *E is a torsionfree right R-module.*

(2) *The set S consists of regular elements of E.*

(3) *If $\phi \in E$ and $\phi * IM = 0$, for an ideal I of R that contains a regular element, then $\phi = 0$.*

Proof. Since S consists of right regular elements of E it is easy to see that (1) and (2) are equivalent.

Now let $\phi \in E$, $s \in S$. Then $\phi * sM = 0$ if and only if $(\phi s) * M = 0$ if and only if $\phi s = 0$. Thus we see that (1) and (3) are equivalent.

2.4. PROPOSITION. *Suppose the set S is a left Ore set for E and that E is a torsionfree right R-module. Then there exists a unique homomorphism ρ: $S^{-1}E \to \mathrm{End}_k(S^{-1}M)$ such that $\rho(\phi)|_M = \phi$ for all $\phi \in E$ and $\rho(r)$ is left multiplication by r whenever $r \in R$. Further, ρ is injective.*

Proof. Clearly it suffices to establish this result with E in place of $S^{-1}E$ as the domain of ρ. If ρ exists, $\phi \in E$, and $t \in S^{-1}M$ then we may pick $p_0 \in S$ such that $p_0 t \in M$ and $q_0 \in S$ and $\phi_0 \in E$ are such that $q_0^{-1}\phi_0 = \phi p_0^{-1}$ (in $S^{-1}E$). With these choices uniqueness follows from the observation that $\rho(\phi) * t = q_0^{-1}\phi_0 * (p_0 t)$.

We may also use this observation as a definition of $\rho(\phi)$ but we must check that it is independent of the choices made. Indeed, suppose $p_1, q_1 \in S$ and $\phi_1 \in E$ are such that $p_1 t \in M$ and $q_1^{-1}\phi_1 = \phi p_1^{-1}$. Then $q_0 \phi_0 p_1^{-1} = (q_0 q_1)\phi(p_0 p_1)^{-1} = q_0 \phi_1 p_0^{-1}$ and so

$$\begin{aligned} q_0^{-1}\phi_0 * (p_0 t) &= (q_0 q_1)^{-1} q_1 \phi_0 p_1^{-1} * (p_1 p_0 t) \\ &= (q_0 q_1)^{-1} q_0 \phi_1 p_0^{-1} * (p_1 p_0 t) \\ &= q_1^{-1}\phi_1 * (p_1 t). \end{aligned}$$

This proves we may define ρ in the appropriate fashion.

Checking that ρ is an injective homomorphism is straightforward.

2.5. Suppose that S is left Ore in E and that E is a torsionfree right R-module. In view of the above proposition we identify $S^{-1}E$ with its image under ρ where appropriate. Given V and W k-submodules of $S^{-1}M$ we may define

$$E(V, W) := \{\theta \in S^{-1}E : \theta * V \subseteq W\}.$$

We also write $E(V) = E(V, V)$. This creates a potential conflict of notation with the existing $E(M, V)$ which may be resolved when (C) obtains.

PROPOSITION. *Suppose S is a left Ore set for E, that E is torsionfree as a right R-module, and that* (C) *holds. Then*

$$\{\theta \in S^{-1}E : \theta * M \subseteq V\} = \{\theta \in E : \theta * M \subseteq V\},$$

for every k-submodule V of M.
In particular, $E = \{\theta \in S^{-1}E : \theta * M \subseteq M\}$.

Proof. Note that it is enough to prove the second statement. Suppose $s^{-1}\theta * M \subseteq M$ where $s \in S$ and $\theta \in E$. Then $\theta * M \subseteq sM$ and therefore $\theta \in E(M, sM) = sE$.

2.6. Suppose that S is left Ore in E and that E is a torsionfree right R-module. There is now a need for some new notation. Let $\hat{\mathscr{I}}(M_k) := \{V$ is a k-submodule of $S^{-1}M : sV \in \mathscr{I}(M_k)$ for some $s \in S\}$, and let $\hat{\mathscr{R}}(M_k)$ denote the sublattice of $\hat{\mathscr{I}}(M_k)$ consisting of those V such that $sV \in \mathscr{R}(M_k)$, for some s $\in$ S.

By localizing at S we create a host of fractional ideals of E. Define $\hat{\mathscr{I}}(E_E) := \{D$ a right E-submodule of $S^{-1}E : sD \in \mathscr{I}(E_E)\}$. We denote the sublattice of $\hat{\mathscr{I}}(E_E)$ consisting of those elements which contain a regular element of R by $\hat{\mathscr{R}}(E_E)$.

Our next result states that the correspondence passes easily to fractional right ideals.

PROPOSITION. *Suppose S is a left Ore set for E, that E is torsionfree as a right R-module, and that* (C) *holds. Then the map $\_ * M$: $\hat{\mathscr{I}}(E_E) \to \hat{\mathscr{I}}(M_k)$ is an injective map. In particular, if $D \in \hat{\mathscr{I}}(E_E)$ then $D = E(M, D * M)$. If, further,* (C′) *holds then $\_ * M$ restricts to a lattice injection $\hat{\mathscr{R}}(E_E) \to \hat{\mathscr{R}}(M_k)$.*

Proof. This follows immediately from (C) on observing that $E(M, sV) = sE(M, V)$ whenever $V \in \hat{\mathscr{I}}(M_k)$ and $s \in S$.

2.7. PROPOSITION. *Suppose S is a left Ore set for E, that E is torsionfree as a right R-module, and that* (C) *holds. Suppose $V, W \in \hat{\mathscr{I}}(E_E) * M$. Then*

$$E(V, W) = \{\theta \in S^{-1}E : \theta E(M, V) \subseteq E(M, W)\}.$$

Proof. Clearly $E(V, W) \subseteq \{\theta \in S^{-1}E : \theta E(M, V) \subseteq E(M, W)\}$. Suppose on the other hand that $\theta \in S^{-1}E$ and $\theta E(M, V) \subseteq E(M, W)$. Then, by Proposition 2.6, evaluating on M we obtain $\theta * V = \theta E(M, V) * M \subseteq E(M, W) * M = W$. This proves the result.

2.8. Now we invoke together the left and right Ore conditions.

LOCALIZATION HYPOTHESIS (L). *S is a left and right Ore set in E and E is a torsionfree right R-module.*

2.9. PROPOSITION. *Suppose that* (C) *and* (L) *hold.*
If $V \in \hat{\mathscr{R}}(E_E) * M$ *then* $\operatorname{End}(E(M,V))_E \cong E(V)$.
If, in addition, $E(M,V)$ *is finitely generated and projective* (*which happens when* R *is a Dedekind domain and* (F) *obtains, for example*) *and* E *is right* (*or left*) *Noetherian, then* $E(V)$ *is right* (*or left*) *Noetherian.*

Proof. This follows from Lemma 2.2, Proposition 2.7, and Lemma 1.8.

2.10. We can immediately deduce our next result, which gives a host of examples of rings Morita equivalent to E.

THEOREM (Examples of Morita Equivalences). *Suppose* (C) *and* (L) *hold. Let* $V \in \hat{\mathscr{R}}(E_E) * M$. *If* $E(M,V)$ *is finitely generated projective* (*which happens when* R *is a Dedekind domain and* (F) *obtains, for example*) *and* $E(M,V)$ *is a generator* (*which happens when* E *is simple, for example*) *then* $E(V)$ *is Morita equivalent to* E *via the mutually dual progenerators* ${}_{E(V)}E(M,V)_E$ *and* ${}_E E(V,M)_{E(V)}$.

2.11. The correspondence (C) classifies the right ideals in $\mathscr{I}(E_E)$. It is natural to ask: When is every finitely generated essential right ideal of E isomorphic to one in $\mathscr{I}(E_E)$? The next result gives one answer to this question. Its proof relies on the correspondence and a modification of the argument of [S, Lemma 4.2].

THEOREM (Classification of Essential Right Ideals). *Suppose that* (C) *and* (L) *hold and that* E *is semiprime Goldie. The following two conditions are equivalent*:

(1) *Every finitely generated essential right ideal of* $S^{-1}E$ *is principal.*

(2) *If* D *is a finitely generated, essential right ideal of* E *then* $D \cong E(M,V)$ *for some* $V \in \mathscr{R}(E_E) * M$.

Proof. Suppose that condition (1) holds and D is a finitely generated, essential right ideal of E. Then $DS^{-1}E = dS^{-1}E$ for some $d \in D$. In fact, d must be a regular element as D is essential. Now D, and therefore $d^{-1}D$, is finitely generated and so we may clear denominators to write

$$d^{-1}D = s^{-1} \sum_{i=1}^{t} d_i E$$

for some $s \in S$ and $d_1, \dots, d_t \in E$. Define

$$D' := \sum_{i=1}^{t} d_i E = s\, d^{-1} D \supseteq sR.$$

By (C), this right ideal is an $E(M,V)$ isomorphic to D.

Conversely, suppose (2) holds and that D is a finitely generated, essential right ideal of $S^{-1}E$. Then there exists a finitely generated essential right ideal D' of E with $D'S^{-1} = D$. Thus, $D \cong E(M,V)S^{-1} = S^{-1}E$, for some $V \in \mathscr{R}(E_E) * M$ and, in particular, D is cyclic as required.

2.12. The previous result takes on its simplest and most appealing form in the following special case.

COROLLARY (Classification of Right Ideals). *Suppose that* (C) *and* (L) *hold. Suppose that E is an Ore domain and that $S^{-1}E$ is a principal right ideal domain. Then every non-zero right ideal of E is isomorphic to one of the form $E(M,V)$, with $V \in \mathscr{I}(E_E) * M$.*

Suppose, further, that (F) *holds. If R is Noetherian then E is right Noetherian. If R is a Dedekind domain then E is hereditary.*

Proof. This is immediate, in view of the observation made in Lemma 1.8 that $E(M,V)$ is always finite generated for R Noetherian, and $E(M,V)$ is always projective for R Dedekind.

2.13. Now we come to our main result, a classification of the rings Morita equivalent to E which have the same uniform dimension.

THEOREM (Classification of Morita Equivalent Rings). *Suppose that* (C) *and* (L) *hold and that E is semiprime Goldie. Further, suppose that every finitely generated essential right ideal of $S^{-1}E$ is principal. If F is a ring Morita equivalent to E and with the same right uniform dimension then $F \cong E(V)$ for some $V \in \mathscr{R}(E_E) * M$.*

Proof. This follows immediately from Theorem 2.11 since if the $E(M,V)$ contain a set of representatives for the isomorphism classes of progenerators with the same uniform dimension as E then their endomorphism rings must contain a set of representatives of the isomorphism classes of Morita equivalent rings with the same uniform dimension as E.

2.14. As usual, this result occurs in its most appealing form in a special case. In fact, it is worth isolating the new hypothesis.

Generically a PRID (G). (a) E is an Ore domain;

(b) $S^{-1}E$ is a principal right ideal domain.

COROLLARY (Classification of Morita Equivalent Domains). *Suppose that* (C), (L), *and* (G) *hold. If F is a domain Morita equivalent to E then $F \cong E(V)$ for some $V \in \mathscr{I}(E_E) * M$.*

3. THE CORRESPONDENCE FOR PROJECTIVE LEFT IDEALS

3.1. Recall from 1.4 that we make the basic hypothesis (B). We suppose *throughout this section* that (L) holds and that R is a (commutative) domain.

We use the notation $\hat{\mathscr{P}}(E_E)$ for the elements of $\hat{\mathscr{I}}(E_E)$ that are finitely generated projective. Recall from Lemma 1.8 that if R is a Dedekind domain and (F) obtains then $\hat{\mathscr{P}}(E_E) = \hat{\mathscr{I}}(E_E)$. We shall use the notation $\hat{\mathscr{I}}({}_E E)$ for $\{D$ a left E-submodule of $S^{-1}E : Ds$ is a left ideal of E that contains a non-zero ideal of R, for some $s \in S\}$ and the notation $\hat{\mathscr{P}}({}_E E)$ for the elements of $\hat{\mathscr{I}}({}_E E)$ which are projective.

Remark. If $D \in \hat{\mathscr{P}}(E_E)$ then $\Phi_D \colon D \otimes_E M \to D * M$ is an isomorphism. In particular, if (C) holds and $\hat{\mathscr{I}}(E_E) = \hat{\mathscr{P}}(E_E)$ then (C′) holds.

Proof. Since D_E is flat the natural map $D \otimes_E M \to D \otimes_E S^{-1}M$ is injective. On the other hand,

$$D \otimes_E S^{-1}M \cong D \otimes_E ES^{-1} \otimes_E M \cong ES^{-1} \otimes_E M \cong S^{-1}M$$

and so Φ_D is injective.

The final statement follows from Theorem 1.3.

3.2. If $D \in \hat{\mathscr{I}}({}_E E)$ then define $D *^{-1} M := \{t \in S^{-1}M : D * t \subseteq M\}$.

THEOREM (Correspondence for Projective Left Ideals). *Suppose that* (C) *holds. There is a bijection* $E(\_, M) \colon \hat{\mathscr{P}}(E_E) * M \to \hat{\mathscr{P}}({}_E E)$. *The inverse map* $\hat{\mathscr{P}}({}_E E) \to \hat{\mathscr{P}}(E_E) * M$ *is defined by* $D \mapsto D *^{-1} M$.

Proof. The left Ore condition for S ensures that $\mathrm{Hom}_E(\_, E)$ vanishes on D'/D, whenever $D \subseteq D'$ are in $\hat{\mathscr{I}}({}_E E)$. Thus, $\mathrm{Hom}_E(D', E) \to \mathrm{Hom}_E(D, E)$ is injective. Since $\mathrm{Hom}_E(Es, E) = s^{-1}E$, for s a unit in $S^{-1}R$, it follows that $\mathrm{Hom}_E(\_, E)$ induces a map $\alpha \colon \hat{\mathscr{P}}({}_E E) \to \hat{\mathscr{P}}(E_E)$. Similarly, $\mathrm{Hom}(\_, E)_E$ induces a map $\beta \colon \hat{\mathscr{P}}(E_E) \to \hat{\mathscr{P}}({}_E E)$. It is easy to see that α and β are inverse. The first part of the result follows immediately from (C) and Proposition 2.6.

For the second claim, observe that if $D \in \hat{\mathscr{P}}({}_E E)$ there is a natural map $\Psi_D \colon D *^{-1} M \to \mathrm{Hom}_E(D, M)$. On the other hand, localization induces a map

$$\mathrm{Hom}_E(D, M) \to \mathrm{Hom}_{S^{-1}E}(S^{-1}D, S^{-1}M)$$

which is injective since D and M have no S-torsion. But $S^{-1}D = S^{-1}E$ and so $\mathrm{Hom}_E(D, M)$ identifies with the k-submodule of $S^{-1}M$ consisting of those t such that $D * t \subseteq M$. Thus Ψ_D is an isomorphism.

There is also a natural map $\mu_D : \mathrm{Hom}_E(D, E) \otimes_E M \to \mathrm{Hom}_E(D, M)$. It is easy to see that μ_D is an isomorphism using the projectivity of D. The result follows.

Remark. One cannot replace $\hat{\mathscr{P}}({}_E E)$ and $\hat{\mathscr{P}}(E_E)$ by $\hat{\mathscr{I}}({}_E E)$ and $\hat{\mathscr{I}}(E_E)$ in the above result. For example, take $E = M = R$ to be a regular Noetherian domain of Krull dimension at least two and m a prime ideal of R with height at least two. Then it is easy to see that $m*^{-1}R = R$.

3.3. Consider the following left handed version of (F).

FILTRATION HYPOTHESIS ON THE LEFT (FL). (a) *Whenever I is a non-zero ideal of R and N is a non-zero subfactor of E/EI then N has a finite chain of submodules*

$$N = N_0 \supset N_1 \supset \cdots \supset N_t = 0$$

such that each factor $N_i/N_{i+1} \cong E/EP_i$, for some $P_i \in \mathrm{Spec}\, R$ containing I.

(b) *E is a flat as a right R-module.*

Remark. Suppose that (FL) holds and that R is Noetherian. Let $D \in \hat{\mathscr{I}}({}_E E)$. Then D is finitely generated.

If, in addition, R is Dedekind domain then D is projective.

3.4. It is also perhaps worth remarking that in the special case that (F) holds, R is a Dedekind domain and E is a Noetherian domain E, then the new condition (FL) (or equivalently, that E/Em is simple for all maximal ideals m of R) holds automatically.

Remark. Suppose that (F) holds, that R is a Dedekind domain, and that E is a Noetherian domain. Then E/Em is simple for all maximal ideals m of R.

Proof. It is well known (and easy to check) that, since E is a hereditary Noetherian domain, $\mathrm{Ext}^1_E(\_, E)$ gives a duality between finite length left and right E-modules. In particular, it maps simples to simples. $m^*/R \cong R/m$ as R-modules and so $\mathrm{Ext}^1_E(m^*E/E, E)$ is a simple left E-module. Computing this module from the resolution $0 \to E \to m^*E \to m^*E/E \to 0$, we clearly obtain E/Em, hence the result.

4. BIMODULE CALCULUS, MORITA THEORY, AND FINITE DIMENSIONAL FACTORS

4.1. Recall that we make the basic hypothesis (B) of 1.4. *Throughout this section* we suppose that (C) and (L) hold, that R is a (commutative) domain, and that E is simple and Noetherian.

We first turn to the Morita Theory. Progenerators will be of the form ${}_{E(W)}E(V,W)_{E(V)}$. The projectivity will follow from Lemma 1.8 (if R is Dedekind) while the generator property will flow from the assumption that E is a simple ring.

4.2. PROPOSITION. $E(V,W)E(U,V) = E(U,W)$ *whenever* $U,V,W \in \hat{\mathscr{P}}(E_E) * M$.

Proof. (1) Note that $E(M,V)E(V,M) = E(V)$ is a consequence of the dual basis lemma and the fact that $E(V,M) = \operatorname{Hom}(E(M,V)_E, E_E)$ and $E(V) = \operatorname{End}(E(M,V))_E$, by Lemma 2.2 and Propositions 2.7 and 2.9.

(2) A slightly more complicated variant is the following:

$$\begin{aligned} E(U,V)E(M,U) &= E(V)E(U,V)E(M,U) \\ &= E(M,V)E(V,M)E(U,V)E(M,U) \\ &= E(M,V)E \\ &= E(M,V). \end{aligned}$$

The second equality is an application of (1) and the third follows from the simplicity of E.

(3) Further, $E(M,V)E(U,M) = E(U,V)E(M,U)E(U,M) = E(U,V)E(U) = E(U,V)$ with the first equality a consequence of (2) and the second a consequence of Theorem 2.10.

(4) Finally, we have

$$\begin{aligned} E(V,W)E(U,V) &= E(M,W)E(V,M)E(M,V)E(U,M) \\ &= E(M,W)EE(U,M) \\ &= E(U,W). \end{aligned}$$

4.3. COROLLARY. *Let U, V, and $W \in \hat{\mathscr{P}}(E_E) * M$. Then the natural map*

$$E(V,W) \otimes_{E(V)} E(U,V) \to E(U,W)$$

is an $E(W)$, $E(U)$-bimodule isomorphism.

Proof. By virtue of the previous result, it is enough to prove that the natural bimodule map

$$\alpha_{U,V,W} : E(V,W) \otimes_{E(V)} E(U,V) \to E(U,W)$$

is injective. Now, $W \subseteq sV$, for some s a unit in $S^{-1}R$. Consider the commutative diagram

$$\begin{array}{ccc} E(V,W) \otimes_{E(V)} E(U,V) & \longrightarrow & E(V,sV) \otimes_{E(V)} E(U,V) \\ \downarrow{\scriptstyle \alpha_{U,V,W}} & & \downarrow{\scriptstyle \alpha_{U,V,sV}} \\ E(U,W) & \longrightarrow & E(U,sV). \end{array}$$

Since $E(V,sV) = sE(V)$, the rightmost vertical map is certainly an isomorphism. So it suffices to note that the top horizontal map is injective. This follows from the fact that ${}_{E(V)}E(U,V)$ is finitely projective (from the proposition and the dual basis lemma) and hence flat.

4.4. COROLLARY. *$E(V)$ is Morita equivalent to $E(W)$ via the progenerator ${}_{E(W)}E(V,W)_{E(V)}$ whenever $V, W \in \hat{\mathscr{P}}(E_E) * M$.*

Proof. The result in the case where V or W is M follows from Theorem 2.10. The general case follows immediately from this case using the last result.

4.5. *Throughout the remainder of this section* we suppose that $\hat{\mathscr{I}}(E_E) = \hat{\mathscr{P}}(E_E)$ and that $\hat{\mathscr{I}}({}_EE) = \hat{\mathscr{P}}({}_EE)$ (which is automatic when hypotheses (F), (FL) hold and R is Dedekind). Note that (C′) holds by Remark 3.1.

DEFINITION. Given $V \in \mathscr{I}(M_k)$, define V^- to be the unique maximal element of $\mathscr{I}(E_E) * M$ contained in V and V^+ to be the unique minimal element of $\mathscr{I}(E_E) * M$ containing V.

Remark. $V^- \subseteq V \subseteq V^+$ with equalities if and only if $V \in \mathscr{I}(E_E) * M$.

4.6. LEMMA. *Let $V \in \mathscr{I}(M_k)$. Then $V^- = E(M,V) * M$ and $V^+ = E(V,M) *^{-1} M$.*

Proof. Note that

$$V \supseteq E(M,V) * M \supseteq E(M,V^-) * M = V^-.$$

This proves the first claim. For the second, note that $V \subseteq V^+ \subseteq E(V,M) *^{-1} M$ and so

$$E(V,M) \supseteq E(V^+,M) \supseteq E(V,M),$$

using Theorem 3.2.

4.7. PROPOSITION. *Suppose that $V, W \in \mathscr{I}(M_k)$. Then*

$$\begin{aligned} E(V^+,W) = E(V^+,W^-) &= E(V,W^-) \\ &\subseteq E(V,W) \\ &\subseteq E(V^+,W^+) \cap E(V^-,W^-). \end{aligned}$$

Proof. First equality:

$$\begin{aligned} E(V^+,W^-) \subseteq E(V^+,W) &= E(V^+,W)E(M,V^+)E(V^+,M) \\ &\subseteq E(M,W)E(V^+,M) \\ &= E(M,W^-)E(V^+,M) \\ &= E(V^+,W^-). \end{aligned}$$

Second equality:

$$\begin{aligned} E(V^+,W^-) \subseteq E(V,W^-) &= E(M,W^-)E(W^-,M)E(V,W^-) \\ &\subseteq E(M,W^-)E(V,M) \\ &= E(M,W^-)E(V^+,M) \\ &= E(V^+,W^-). \end{aligned}$$

Clearly $E(V^+,W^-) \subseteq E(V,W)$.

The second inclusion comes from the following observations.

$$\begin{aligned} E(V,W) &\subseteq \left\{\theta \in S^{-1}E : \theta E(M,V) \subseteq E(M,W)\right\} \\ &= \left\{\theta \in S^{-1}E : \theta E(M,V^-) \subseteq E(M,W^-)\right\} \\ &= E(V^-,W^-). \\ E(V,W) &\subseteq \left\{\theta \in S^{-1}E : E(W,M)\theta \subseteq E(V,M)\right\} \\ &= \left\{\theta \in S^{-1}E : E(W^+,M)\theta \subseteq E(V^+,M)\right\} \\ &= E(V^+,W^+). \end{aligned}$$

4.8. Next we come to our main result of this section showing the existence of a unique minimal essential ideal of $E(V)$. First, some notation. We write $J(V,W)$ for the bimodule $E(V^+,W^-)$.

THEOREM. *$E(V,W)$ has a unique minimal $E(W)$–$E(V)$-subbimodule containing a regular element of $S^{-1}E$ and that bimodule is $J(V,W)$. Further, there are canonical injections*

$$E(V,W)/J(V,W) \to \mathrm{Hom}_k(V/V^-,W/W^-)$$

and

$$E(V,W)/J(V,W) \to \mathrm{Hom}_k(V^+/V,W^+/W).$$

Proof. Suppose D is a non-zero $E(W)$–$E(V)$-subbimodule of $E(V,W)$ that contains a regular element of $S^{-1}E$. Then D contains

$E(W, W^-)DE(V^+, V)$ which is an $E(W^-)$-$E(V^+)$-subbimodule of $E(V^+, V^-)$ containing a regular element of $S^{-1}E$. Therefore

$$J(V,W) = E(V^+,W^-) = E(W,W^-)DE(V^+,V) \\ \subseteq D.$$

Thus $J(V,W)$ is the minimal non-zero subbimodule of $E(V,W)$.

If $\phi \in E(V,W)$ then $\phi \in E(V^-,W^-)$ so there is a canonical map θ: $E(V, W) \to \mathrm{Hom}_k(V/V^-, W/W^-)$. This map has kernel $\{\phi \in E(V,W) : \phi * V \subseteq W^-\} = E(V,W) \cap E(V,W^-) = E(V,W^-)$. Thus there is a canonical map

$$E(V,W)/J(V,W) \to \mathrm{Hom}_k(V/V^-,W/W^-).$$

The other canonical map is similarly derived.

Remark. The above result was proved in the special case $E = \mathscr{D}(R)$ in [CH2] using completion arguments. The more general framework of this paper has made it easier to see that this result obtains for purely ring-theoretic reasons.

4.9. Finally, we give a criterion for $E(V)$ to be Noetherian.

THEOREM. *Suppose that* (F) *and* (FL) *hold. Suppose that k is a field and that R/m is finite-dimensional over k, for all maximal ideals of R. Suppose that M is a finitely generated R-module and that R is a Dedekind domain.*

Then $E(V)$ is Noetherian, for $V \in \mathscr{I}(M_k)$.

Proof. We have the containments

$$E(V^+,V^-) \subseteq E(V) \subseteq E(V^+) \cap E(V^-).$$

Note that $E(V^+)$ and $E(V^-)$ are both Noetherian, as they are Morita equivalent to E, by Theorem 2.10. Note also that $E(V^{\pm})/E(V^+, V^-)$ has finite length as a right $E(V^+)$ module (resp. left $E(V^-)$-module). Further, if N is a simple $E(V^{\pm})$-module subfactor of $E(V^{\pm})/E(V^+, V^-)$ then $\mathrm{End}\, N$ is finite-dimensional over k. For, these two statements are Morita invariant and so can be verified for $E(M, V^+)/E(M, V^-)$ and $E(V^-, M)/E(V^+, M)$, respectively (see Lemma 1.6 and Remark 1.7).

The result follows by appealing to [RS, Proposition 1(a)].

5. EXAMPLES

5.1. Recall that we always make the hypothesis (B).

In this final section we examine various examples for which (C), (C′) (often the stronger statement (F)), (L), and (G) all hold. As a consequence, all the most important results of this paper are applicable for these examples. Specifically, we are able to classify the right ideals of E and the domain Morita equivalent to E.

Our first examples are rings of differential operators on smooth affine curves. Next we consider simple skew Laurent and differential polynomial rings over commutative Noetherian domains. Then, we treat the so-called "quantum" Weyl algebra. Finally, we consider an idealizer example and a ring of invariants.

5.2. In many of our main examples, $M = R$ and R is a Dedekind domain so we suppose this throughout the remainder of this subsection. Suppose, further, that k is an algebraically closed field and that R is *k-rational*, that is, $R/m \cong k$, as k-algebras for every maximal ideal m of R. Finally, assume that (C′) holds. To understand the possibilities for the shape of $\mathscr{I}(E_E) * R$ it is quite instructive to consider a rather special sublattice, namely those elements of $\mathscr{I}(E_E) * R$ that contain mm', for maximal ideals m, m' of R. There are three possibilities for the shape depending on the length two module $E/mm'E$. It is either

(uniserial case) $\{mm', m, R\}$, or

(completely reducible, isotypic case) the set of k-submodules of R that contain mm' which bijects to $\mathbb{P}^1_k$, or

(completely reducible, non-isotypic case) $\{mm', m, m', R\}$.

To see how these possibilities arise, note that $E/mm'E$ is either uniserial or completely reducible. If $E/mm'E$ is uniserial we must have $m = m'$ and the unique composition series is $0 \subset mE/m^2E \subset E/m^2E$. On the other hand, if $E/mm'E$ is completely reducible it is isomorphic to $E/mE \oplus E/m'E$. Thus, which of the remaining two possibilities occurs depends on whether $E/mE \cong E/m'E$, or not.

5.3. The results in Sections 1–4 of this paper are a generalization of the results of [CH1] and [CH2]. Naturally enough, the examples in those papers, differential operator rings, provide our first main examples.

Suppose that k is an algebraically closed field of characteristic zero and that R is a k-rational Dedekind domain which is a localization of a finitely generated k-algebra. Let $E = \mathscr{D}(R)$ be the ring of k-linear differential operators on R. Thus, $\mathscr{D}(R)$ is the k-subalgebra of $\operatorname{End}_k R$ generated by R and the k-linear derivations of R. Then E satisfies (F) ($\Rightarrow$ (C′)), (L), and (G). Further, E is a simple Noetherian domain (see [CH1, Theorem 1.12] and [MR, Theorems 15.1.25, 15.3.7], for example). In particular, all the results of Sections 1–4 are applicable to E.

Further, in this case the lattice $\mathscr{I}(E_E) * R$ can be easily described. It has an extremely regular structure. A subspace $V \in \mathscr{I}(M_k)$ is said to be *m-primary*, for a maximal ideal m of R, if V contains some power of m. V

is said to be *primary decomposable* if it is a finite intersection of primary subspaces of R. Remarkably, this simple notion is enough to describe the lattice $\mathscr{I}(E_E)$. For, $\mathscr{I}(E_E) * R$ consists of exactly the primary decomposable subspaces (see [CH1]). In [CH2] and [CH3] one discovers more detailed results about the finite-dimensional factor rings of Section 4. For example, suppose that R has infinitely many maximal ideals. Then, as $V \in \mathscr{I}(M_k)$ varies, the factors $E(V)/J(V)$, of Section 4, run through all finite-dimensional k-algebras.

Remark. The assumption that $\dim R = 1$, here, is essential as can be seen by the maximal right ideal $D = \partial/\partial x_1 \mathscr{D}(k[x_1, \dots, x_n]) + \sum_{i=2}^{n} x_i \mathscr{D}(k[x_1, \dots, x_n])$ of $\mathscr{D}(k[x_1, \dots, x_n])$, where $n \geq 2$. For this right ideal clearly has $D * k[x_1, \dots, x_n] = k[x_1, \dots, x_n]$.

5.4. Suppose in this subsection that R is a Noetherian (non-Artinian) domain and that σ is a k-algebra automorphism of R which leaves no proper, non-zero ideal of R invariant. Let E be the skew Laurent polynomial ring $R[x, x^{-1}; \sigma]$ which identifies with $R[\sigma, \sigma^{-1}] \subset \operatorname{End}_k R$. Then it is well known that E is a simple Noetherian domain and satisfies hypotheses (F) ($\Rightarrow$ (C$'$)), (L), and (G). For this, see [MR, Theorem 1.8.5, Lemma 6.9.16, Corollary 11.2.13]. Thus, we see that every non-zero right ideal of E is isomorphic to one of the form $E(R, V)$, for some $V \in \mathscr{I}(E_E) * R$. Further, any domain Morita equivalent to E is isomorphic to some $E(V)$, with $V \in \mathscr{I}(E_E) * R$. If, in addition, R is Dedekind then each $E(V)$, with $V \in \mathscr{I}(E_E) * R$, will be Morita equivalent to E.

We consider the shape of the lattice $\mathscr{I}(E_E) * R$ in the case when R is Dedekind. So suppose that R is Dedekind in the sequel. It is much more complicated than in the differential operator case just discussed. There are two main differences. If m is a maximal ideal of R then $\mathscr{D}(R)/m^n\mathscr{D}(R)$ is completely reducible, and it is isomorphic to n copies of $\mathscr{D}(R)/m\mathscr{D}(R)$ (see [CH1, Theorem 1.12], for example). By way of contrast:

LEMMA. *If m is a maximal ideal of R and $n > 0$ then E/m^nE is a uniserial module.*

Proof. We proceed by induction on n, the case $n = 1$ being obvious. So suppose that $n > 1$. Let N be a simple submodule of E/m^nE and choose $f \in E$ with minimal support (considered as a polynomial in x) such that the image of f generates N. If $f = hx^t$, with $h \in R$ then it is not hard to see that $N = m^{n-1}E/m^nE$. By induction, $E/m^{n-1}E$ has a unique composition series and so we are done.

Suppose then, without loss of generality, that $f = \sum_{i=s}^{t} f_i x^i$ with $f_i \in m^{n_i} \setminus m^{n_i+1}$, and that each n_i is either $< n$ or equal to ∞ (which is the case when $f_i = 0$). Further, we may as well suppose that there are at least two non-infinity values, say n_0 and n_t. Choose $y \in \sigma^{-t} * m^{n-n_t} \setminus m$. Then

fy has smaller support than f and its image in N is clearly still non-zero. Hence, the result.

As an immediate consequence of the lemma we see that the elements of $\mathscr{I}(E_E) * R$ which contain m^n are simply $m^i : 0 \le i \le n$. Whereas, *every* k-submodule of R containing m^n is in $\mathscr{I}(\mathscr{D}(R)_{\mathscr{D}(R)}) * R$. So it appears that we obtain "fewer" subspaces for E than for $\mathscr{D}(R)$.

The second difference with $\mathscr{D}(R)$ arises as follows. Let $m \ne m'$ be distinct maximal ideals of R. Then $\mathscr{D}(R)/m\mathscr{D}(R) \not\cong \mathscr{D}(R)/m'\mathscr{D}(R)$ by [CH1, Remark 1.15]. On the other hand, if we now suppose that $m \ne m'$ are in the same $\langle \sigma \rangle$-orbit then we have $E/mE \cong E/m'E$ (the isomorphism is induced by left multiplication by the appropriate power of σ). As a consequence of this we see, for example, that the elements of $\mathscr{I}(E_E) * R$ containing mm' biject to $\mathbb{P}^1$ (at least if k is an algebraically closed field, since they correspond to the submodules of $E/mm'E$ which is two copies of the same simple module). On the other hand, the elements of $\mathscr{I}(\mathscr{D}(R)_{\mathscr{D}(R)}) * R$ containing mm' are simply mm', m, m', and R. So it appears that we obtain "more" subspaces for E than for $\mathscr{D}(R)$!

To give a more specific example of the above take $R = k[y, y^{-1}]$, where k is an algebraically closed field and σ is the automorphism given by $y \mapsto \lambda y$, for $\lambda \in k^*$ not a root of unity. We can describe $\mathscr{I}(E_E) * R$ completely.

PROPOSITION. (a) *If $V \in \mathscr{I}(E_E) * R$ then $V = \bigcap_{\Sigma \in \mathrm{Max}\, R/\langle \sigma \rangle} V_\Sigma$, where $V_\Sigma \in \mathscr{I}(E_E) * R$ and contains a non-zero ideal of R with minimal primes in Σ, and all but finitely many $V_\Sigma = R$.*

(b) *Let $m = (y - \alpha)R$ be a maximal ideal of R and $I = \sigma^{a_1}(m^{t_1}) \cdots \sigma^{a_n}(m^{t_n})$, for positive integers $t_1, \dots, t_n$ and distinct integers $a_1, \dots, a_n$. Let π_r and $\iota_s : 1 \le r, s \le n$ denote the standard projections and injections associated with the decomposition $R/I \cong \oplus R/\sigma^{a_r}(m^{t_r})$ and let π denote the natural map $R \to R/I$. Let $\mathscr{V}(I)$ denote the sublattice of $\mathscr{I}(E_E) * R$ consisting of those k-subspaces in $\mathscr{I}(E_E) * R$ which contain I. Then $\mathscr{V}(I)$ is isomorphic (via π) to*

$$\pi\mathscr{V}(I) = \{\pi I\} \cup \bigcup_{i=1}^{n} G\pi\mathscr{V}\big(\sigma^{a_1}(m^{t_1}) \cdots \sigma^{a_{i-1}}(m^{t_{i-1}})\sigma^{a_i}(m^{t_i - 1}) \times \sigma^{a_{i+1}}(m^{t_{i+1}}) \cdots \sigma^{a_n}(m^{t_n})\big)$$

where G denotes the group of units of the subalgebra

$$\sum_{1 \le i, j \le n} \iota_j \left(\sum_{0 \le r \le \min\{t_i, t_j\}} k\sigma^{a_j}\big((y - \alpha)^{t_j - r}\big)\sigma^{a_j - a_i} \right) \pi_i$$

of $\mathrm{End}_k(R/I)$.

Proof. (a) If $\Sigma \in \operatorname{Max} R/\langle\sigma\rangle$ is an orbit of maximal ideals let S_Σ denote the multiplicatively closed subset $\bigcap_{m\in\Sigma}(R\setminus m)$ of R. It is easy to see that S_Σ is an Ore set of E, since it is σ-stable. Further, if D is a right ideal of then $D=\bigcap_{\Sigma\in\operatorname{Max} R/\langle\sigma\rangle}(E\cap DS_\Sigma^{-1})$. It follows quickly that if $V\in\mathscr{I}(E_E)*R$ then $V=\bigcap_\Sigma V_\Sigma$, where each $V_\Sigma\in\mathscr{I}(E_E)*R$, all but finitely many V_Σ equal R, and V_Σ contains an ideal of R each of whose minimal primes lies in Σ.

In order to prove the second statement we first show that the image of $\operatorname{End}(E/IE)_E$ under the natural injection $\operatorname{End}(E/IE)_E\to\operatorname{End}_k(R/I)$ is the subalgebra

$$\sum_{1\le i,j\le n}\iota_j\left(\sum_{0\le r\le\min\{t_i,t_j\}}k\sigma^{a_j}\left((y-\alpha)^{t_j-r}\right)\sigma^{a_j-a_i}\right)\pi_i.$$

It is clearly enough to show that

$$\operatorname{Hom}(E/m^pE,E/m^qE)=\sum_{0\le r\le\min\{p,q\}}k(y-\alpha)^{q-p+r},$$

for $p,q\ge 1$. Let $\mu\in\operatorname{Hom}(E/m^pE,E/m^qE)$. It is enough to show that if μ is injective and $p=q$ then μ is given by scalar multiplication. By induction, we can assume that μ induces the identity map $mE/m^pE\to mE/m^pE$. Let $\bar\mu:(E/m^pE)/(mE/m^pE)\to(E/m^pE)/(mE/m^pE)$ be the induced map. By induction $\bar\mu$ is given by multiplication by a scalar, β, say. Now $\mu(1+m^pE)=(\beta+(y-\alpha)\theta)+m^pE$, for some $\theta\in E$. Thus, $\mu((y-\alpha)+m^pE)=(\beta(y-\alpha)+(y-\alpha)\theta(y-\alpha))+m^pE=(y-\alpha)+m^pE$, where the last equality is because μ induces the identity map on its unique length $p-1$ submodule. It follows that there exists $\phi\in E$ such that

$$\beta(y-\alpha)+(y-\alpha)\theta(y-\alpha)=(y-\alpha)+(y-\alpha)^p\phi.$$

Cancelling, this gives $\beta-1\in E(y-\alpha)+(y-\alpha)^{p-1}E$. But now an easy computation shows that $E=k\oplus(Em+m^{p-1}E)$, as $p-1\ge 1$.

Next, let S be a simple submodule of E/IE and identify the latter module with $\bigoplus_i E/\sigma^{a_i}(m^{t_i})E$. We claim that there exists an automorphism ϕ of E/IE such that $\phi(S)=\sigma^{a_s}(m^{t_s-1})E/\sigma^{a_s}(m^{t_s})E$, for some $1\le s\le n$. In order to prove this note that the socle of E/IE is simply $\bigoplus\sigma^{a_i}(m^{t_i-1})E/\sigma^{a_i}(m^{t_i})E$. Since the group of automorphisms of the socle acts transitively on the simple submodules of the socle, and $\operatorname{Aut} E/mE=k$, it follows that $S=(\kappa_1\sigma^{a_1}(y-\alpha)^{t_1-1},\dots,\kappa_n\sigma^{a_n}(y-\alpha)^{t_n-1})E$, for some $\kappa_1,\dots,\kappa_n\in k$, not all zero. It is now easy to see (in view of the description of $\operatorname{End} E/IE$ given above) that there exists such a ϕ. Indeed, if we suppose (harmlessly) that $t_1\le\cdots\le t_n$, then $s:=\min\{i:\kappa_i\ne 0\}$.

Now consider a submodule N of E/IE. We can associate a subspace $N \cdot (R/I)$ of R/I to N by defining $N \cdot (R/I) := (D * R)/I$, where D is the unique right ideal of E containing IE with $N = D/IE$. Now if ψ is any endomorphism of the E-module E/IE let $\hat{\psi}$ denote the corresponding k-linear endomorphism of R/I. One checks easily that $\psi(N) \cdot (R/I) = \hat{\psi}(N \cdot (R/I))$.

Now let D be some right ideal of E with $D \supset IE$. Choose $D \supseteq D' \supset IE$, with D'/IE simple. As we have seen, there exists an automorphism ϕ of E/IE such that $\phi(D'/IE) = I'E/IE$, where I' is ideal of R containing I and I'/I is a simple R-module. Note that $\hat{\phi} \in G$. Further,

$$\begin{aligned} \hat{\phi}\pi D * R &= \phi(D/IE) \cdot (R/I) \supseteq \phi(D'/IE) \cdot (R/I) \\ &= (I'E/IE) \cdot (R/I) = \pi I' \end{aligned}$$

and so $\pi D * R \in G\pi\mathscr{V}(I')$.

Let us illustrate the results of Section 4 with an example. Let $V = k + (y-1)(\lambda y - 1)k[y, y^{-1}]$ and $W = k + (y-1)^2 k[y, y^{-1}]$. Then $V \in \mathscr{I}(E_E) * R$ but W is not. In fact, $W^- = (y-1)^2 k[y, y^{-1}]$ and $W^+ = k[y, y^{-1}]$ and so $E(W)/J(W) \cong k$. It follows from this that $E(W) = k + (y-1)^2 E$. Amusingly, exactly the opposite behaviour is exhibited by V and W if we replace E by $\mathscr{D}(k[y, y^{-1}])$!

We complete this subsection by giving some further examples of (R, σ) satisfying the conditions of the first paragraph of this subsection. For example, one can take $R = k[y]$, where k is a field of characteristic zero and σ is the automorphism given by $y \mapsto y + a$, for some $0 \neq a \in k$. Note that $E = k[y, \sigma^{\pm 1}] \cong \mathscr{D}(k[y, y^{-1}])$ and so this ring can be studied using the results of this paper through two different representations.

Let us give a slightly more exotic example. Let X be an elliptic curve over an algebraically closed field k. Choose $p \in X$ a point of infinite order in the group law on X and let $\tau_p \colon X \to X$ be translation (in the group law) by p. Take any subset $\varnothing \subsetneq Z \subsetneq X$ for which $\tau_p Z = Z$; for example, one could take $Z = \mathbb{Z}p$. Then take

$$R = \bigcap_{z \in Z} \mathscr{O}_{X,z}.$$

It is easy to see that the automorphism σ of R induced by τ_p leaves stable no non-zero, proper ideals.

An example for which R is not Dedekind is given by $R = k[y_1^{\pm 1}, \ldots, y_n^{\pm 1}]$, for k a field, and σ the k-algebra automorphism with $\sigma * y_i = \lambda_i y_i$, where $\lambda_1, \ldots, \lambda_n$ generate a subgroup of k^* with rank n. See [MR, 1.8.7].

5.5. Let R be a commutative (non-Artinian) Noetherian $\mathbb{Q}$-algebra which is a domain and let k be a commutative subring of R. Let δ be k-linear derivation of R that leaves stable no proper, non-zero ideal of R. Let E be the skew polynomial ring $R[x;\delta]$ which identifies with $R[\delta] \subset \operatorname{End}_k R$. Then it is well known that E is a simple Noetherian domain and satisfies (F) ($\Rightarrow$ (C′)), (L), and (G). For this, see [MR, Theorem 1.8.4, Lemma 6.9.16, Corollary 12.2.12]. Thus, we see that every non-zero right ideal of E is isomorphic to one of the form $E(R,V)$, for some $V \in \mathscr{I}(E_E) * R$. Further, any domain Morita equivalent to E is isomorphic to some $E(V)$, with $V \in \mathscr{I}(E_E) * R$. If, further, R is Dedekind then each $E(V)$, with $V \in \mathscr{I}(E_E) * R$, will be Morita equivalent to E.

These results are used in [H] (in the case k is algebraically closed) to compute $\operatorname{Pic}_k R[x;\delta]$ and to classify the domains S which have a faithful action by k-algebra automorphisms of C_p, the cyclic group of order p, such that $S^{C_p} = R[x;\delta]$.

A specific example of $R[x;\delta]$ is the one found by Bergman and Archer (see [J]). Namely, $R = k[x_1, \ldots, x_n]$ and $\delta = \partial/\partial x_1 + \sum_{i=2}^{n}(1 + x_i x_{i-1})\partial/\partial x_i$.

5.6. Let k be an algebraically closed field of characteristic zero and $R = k[y]$. Let $q \in k^*$ be of infinite order and let σ be the k-algebra automorphism of R with $\sigma * y = qy$. Let ∂ be the σ-derivation of R with $\partial * y = 1$. Finally, set $E = k[y, \partial, \sigma, \sigma^{-1}] \subset \operatorname{End}_k R$. In the literature E (or sometimes its subalgebra $k[y,\partial]$) is called the "quantum" or "quantized" Weyl algebra.

PROPOSITION. *E is a simple Noetherian domain and satisfies* (F) (*and hence* (C′)), (L), *and* (G).

If $V \in \mathscr{I}(E_E) * R$ then $V = \bigcap_{\Sigma \in \operatorname{Max} R/\langle\sigma\rangle} V_\Sigma$, where V_Σ is in $\mathscr{I}(E_E) * R$ and contains a non-zero ideal of R whose minimal primes are in Σ, and all but finitely many $V_\Sigma = R$.

Every k-subspace of R containing $y^n R$ lies in $\mathscr{I}(E_E) * R$. On the other hand, let $m = (y-\alpha)R$ be a maximal ideal of R, with $\alpha \neq 0$, and $I = \sigma^{a_1}(m^{t_1}) \cdots \sigma^{a_n}(m^{t_n})$, for positive integers $t_1, \ldots, t_n$ and distinct integers $a_1, \ldots, a_n$. Let π_r and ι_s denote the standard projections and injections associated with the decomposition $R/I \cong \bigoplus R/\sigma^{a_r}(m^{t_r})$ and let π denote the natural map $R \to R/I$. Let $\mathscr{V}(I)$ denote the sublattice of $\mathscr{I}(E_E) * R$ consisting of those k-subspaces in $\mathscr{I}(E_E) * R$ which contain I. Then $\mathscr{I}(I)$ is isomorphic (*via* π) *to*

$$\pi\mathscr{V}(I) = \{\pi I\} \cup \bigcup_{i=1}^{n} G\pi\mathscr{V}\big(\sigma^{a_1}(m^{t_1}) \cdots \sigma^{a_{i-1}}(m^{t_{i-1}})\sigma^{a_i}(m^{t_i - 1}) \times \sigma^{a_{i+1}}(m^{t_{i+1}}) \cdots \sigma^{a_n}(m^{t_n})\big)$$

where G denotes the group of units of the subalgebra

$$\sum_{1\le i,j\le n} \iota_j \left(\sum_{0\le r\le \min\{t_i,t_j\}} k\sigma^{a_j}\big((y-\alpha)^{t_j-r}\big)\sigma^{a_j-a_i}\right)\pi_i$$

of $\operatorname{End}_k(R/I)$.

Proof. First note that $\sigma = (q-1)y\partial + 1$. Thus, inverting the Ore set $T = \{y^n : n \ge 0\}$ of E we obtain $k[y^{\pm 1}, \sigma^{\pm 1}]$ which we treated in 5.4, above. This demonstrates the truth (L) and (G) immediately. The equation $\partial y = qy\partial + 1$ together with the simplicity of $T^{-1}E$ quickly yields that E is simple. Now if $m \neq yR$ is a maximal ideal of R and $f \in E \setminus mE$ then $f \notin mET^{-1}$. Thus, $mET^{-1} + fET^{-1} = ET^{-1}$. It follows swiftly that mE is a maximal right ideal of E. On the other hand, suppose that $m = yE$. Choose $f \in E \setminus mE$. Since $\partial\sigma = q\sigma\partial$ then fE may be generated by an element of $k[\partial]$. It is now easy to show that $mE + fE = E$, using induction on degree in ∂. This establishes (F) ($\Rightarrow$ (C$'$)).

The statements of the proposition about the shape of the lattice $\mathscr{I}(E_E) * R$ result from the following facts. If $\Sigma \in \operatorname{Max} R$ then $S_\Sigma = \bigcap_{m\in\Sigma}(R \setminus m)$ is an Ore set of E. This proves the first claim about the shape.

Now E/y^nE is completely reducible. To see this, compute $\operatorname{Ext}^1_E(E/yE, E/yE)$. But $yE + Ey = E$, as is easily checked. Thus, one can appeal to the method of [MR2, Sect. 1] to see that this extension group is zero. Thus, $\operatorname{End} E/y^nE = \operatorname{End}_k R/y^nR$. On the other hand, if I is a non-zero ideal each of whose minimal primes lies in $\Sigma \neq \{y\}$ then one can easily see that there is a lattice isomorphism $\mathscr{L}(E/IE)_E \to \mathscr{L}(ET^{-1}/IET^{-1})_{ET^{-1}}$ and that

$$\operatorname{End}(E/IE)_E \cong \operatorname{End}\big(ET^{-1}/IET^{-1}\big)_{ET^{-1}}.$$

Thus, we can appeal to our existing results for the ring $ET^{-1} = k[y, y^{-1}, \sigma, \sigma^{-1}]$.

5.7. In order to give some more examples we prove a criterion for (C$'$) that holds in some cases where (F) fails.

PROPOSITION. *Suppose that R is a Dedekind domain, that E is a hereditary Noetherian domain, and that* (L) *holds. Further, suppose that whenever m is a maximal ideal of R either*:

(1) *mE is a maximal right ideal of E, or*

(2) $S \otimes_E M \neq 0$, *for every simple subfactor of* E/mE.

Then (C$'$) *holds.*

Proof. We apply Theorem 1.3 with $\mathscr{L} = \mathscr{I}(E_E)$. By virtue of Remark 3.1 we know that (1.2.2′), and hence (1.2.2), is satisfied. Thus it remains to show that $D'/D \otimes_E M \neq 0$, whenever $D', D \in \mathscr{I}(E_E)$ with $D' \supsetneqq D$. This, in turn, will follow if $S \otimes_E M \neq 0$ whenever S is a simple subfactor of E/mE, for m a maximal ideal of R. If mE is maximal this is clear, as ${}_RM$ is faithfully flat. If mE is not maximal it follows by hypothesis (2).

5.8. Consider the following two examples.

(1) Let k be a field of characteristic zero, let $R = k[y]$, let $M = yk[y] \oplus k[y]$, and let $E = k + yk[y, \partial/\partial y]$.

(2) Let k be a field, let $R = k[y, y^{-1}]$, and let σ be the k-algebra automorphism of R with $y \mapsto \lambda y$, for some $\lambda \in k^*$ not a root of unity. Let $M = (y - \alpha)k[y, y^{-1}] \oplus k[y, y^{-1}]$, for $\alpha \in k^*$, and let $E = k + (y - \alpha)k[y, y^{-1}, \sigma, \sigma^{-1}]$.

Then, in either case, E is a hereditary Noetherian domain but does not satisfy (F). However, it does satisfy (C′), (L), and (G). In particular, $\_ * M$: $\mathscr{I}(E_E) \to \mathscr{I}(M_k)$ is a lattice injection, $D = E(M, D * M)$ for $D \in \mathscr{I}(E_E)$, and every non-zero right ideal of E is isomorphic to one in $\mathscr{I}(E_E)$. Further, if E is a domain Morita equivalent to E then $E \cong E(V)$, for some $V \in \mathscr{I}(E_E) * M$. On the other hand, if $V \in \mathscr{I}(E_E) * M$ then $E(V)$ is Morita equivalent to E or to A_1 (resp., $k[y, y^{-1}, \sigma, \sigma^{-1}]$) depending on whether it is simple or not; or equivalently, whether $E(M, V)$ is a generator or not.

Proof. Note that (L) and (G) are obvious. We apply the previous proposition to show that (C′) holds. Let $m_0 = yk[y]$ in case (1) and $(y - \alpha)k[y, y^{-1}]$ in case (2). Thus, $M = m_0 \oplus R$. let $E' = k[y, \partial/\partial y]$ in case (1) and $k[y, y^{-1}, \sigma, \sigma^{-1}]$ in case (2).

Note that E/m_0E has length two. For $E \supset m_0E' \supset m_0E$ and $E/m_0E' \cong k$ is simple, $m_0E'/m_0E \cong E'/E$ is simple (by [MR, Proposition 5.5.5]). We must show that $S \otimes_E M \neq 0$, for these simple subfactors. This is clear for $S = E/m_0E'$, as $m_0E' * (m_0 \oplus R) = m_0 \oplus m_0$.

Now, in case (1), $E' = \partial/\partial yE + E$. Consider $D = yE + (y\partial/\partial y - 1)E$. Evaluating on M one sees that $E \supsetneqq D \supsetneqq yE$. It is not hard to see from this that $E'/E \cong E/D$. In particular, $(E'/E) \otimes_E M \neq 0$. In case (2), $E' = \sigma E + E$. Consider $D' = (\lambda^{-1}y - \alpha)E + (y - \alpha)\sigma^{-1}E$. Evaluating on M one checks that $E \supsetneqq D' \supsetneqq (\lambda^{-1}y - \alpha)E$ and, thus, $E'/E \cong E/D'$. Again, it follows that $(E'/E) \otimes_E M \neq 0$.

Let m be some maximal ideal of R with $m \neq m_0$. It is easily seen that the natural map $E/mE \to E'/mE'$ is injective. Now, by virtue of [MR, Proposition 5.5.5], E'/mE' is simple as an E-module, in case (1). Thus E/mE is a simple E-module. Similarly, in case (2), when $m \notin \langle \sigma \rangle m_0$,

E/mE is simple. Finally, in case (2), with $m \in \langle \sigma \rangle m_0 \setminus \{m_0\}$, E/mE embeds in $E'/mE' \cong E'/m_0E'$. By [MR, Proposition 5.5.5], E'/m_0E' has a unique composition series of length two, namely: $m_0E' \subset E \subset E'$. But we have already established that $(E/m_0E') \otimes_E M \neq 0$ and $(E'/E) \otimes_E M \neq 0$. The result follows from the proposition.

Remark. It is easy to see that the above calculations fail if we take $M = R$ or $M = m_0$.

5.9. Finally, another example for which (F) fails, but for which (C′), (L), and (G) hold, so that all the main results of this paper are applicable. Let k be a field of characteristic zero, $R = k[y^n]$, $M = k[y]$ and $E = k[y^n, y\partial/\partial y, \partial^n/\partial y^n]$, for some $n \geq 2$. It is well known that E is a simple Noetherian domain.

PROPOSITION. *The map $\_ * M$: $\mathscr{I}(E_E) \to \mathscr{I}(M_k)$ is an injective lattice map, and every non-zero right ideal of E is isomorphic to one in $\mathscr{I}(E_E)$. If $V \in \mathscr{I}(E_E) * M$ then $E(V)$ is Morita equivalent to E, and conversely, every domain Morita equivalent to E is isomorphic to some $E(V)$, with $V \in \mathscr{I}(E_E) * M$.*

Proof. Note first that E satisfies (L) and (G); localizing at S one clearly obtains $\mathscr{D}(k(y^n))$.

We will apply Proposition 5.7. If $m \neq y^n k[y^n]$ then mE is a maximal right ideal of E. To see this note that $T = \{y^{in} : i \geq 0\}$ is a left and right Ore set of E and that the localization $T^{-1}E$ is $\mathscr{D}(k[y^n, y^{-n}])$. We saw above that mET^{-1} is a maximal right ideal of ET^{-1}. It follows easily that mE is a maximal right ideal of E.

Thus, the only possible difficulty occurs with $m = y^n k[y^n]$. Now E/y^nE is completely reducible of length n with simple subfactors $E/y^nE + (y\partial/\partial y - i)E$, for $0 \leq i \leq n-1$. To prove this, for $0 \leq i \leq n-1$, set $D_i := y^nE + (y\partial/\partial y - i)E$. Then $M/D_i * M$ is one dimensional generated by the class of y^i. In particular, D_i is proper. On the other hand, $E/D_i = k[\partial^n/\partial y^n] + D_i/D_i$ is a $k[\partial^n/\partial y^n]$-module. Thus any proper factor of E/D_i is finite-dimensional over k. Since E is a simple ring this tells us that E/D_i is simple. Finally, set $N_i := y^nE + \prod_{0 \leq j \neq i \leq n-1}(y\partial/\partial y - j)E/y^nE$. It is easy to see that $N_i \cong E/D_i$ and that $E/y^nE = \sum N_i$, hence E/y^nE is completely reducible of length n as required. The result follows from Proposition 5.7.

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